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1 Abstract

In this work, we investigate spin-gravity coupling in a plane gravitational wave spacetime and explicitly derive the Mathisson–Papapetrou–Dixon (MPD) equations for this metric. The motion of a classical spinning particle in curved spacetime is one of the most fascinating problems in general relativity. The dynamics of such a particle’s motion are studied using the MPD equations within the pole-dipole approximation. These equations indicate that compared to the case of a spinless particle, there is a difference in the particle’s motion, and it is possible for the particle to deviate from the geodesic path. In fact, due to an additional term arising from the coupling of spin and spacetime curvature, acting like a force, the particle deviates from the geodesic trajectory.

In this work, we focus on investigating spin-gravity coupling in the metric corresponding to a plane gravitational wave in the Rosen coordinates. The reason we are interested in Rosen coordinates is that their geometric and mathematical properties greatly simplify the calculations. Numerous studies exist on solving the MPD equations to first order in spin in systems whose geometry is rich in symmetries. For example, articles related to solving these equations in the Schwarzschild metric or the Kerr metric. However, there are a few articles concerning the analytical solution of these equations for a general plane gravitational wave spacetime such as [22], [21], [24], and [23], which we will refer to frequently.

The main focus is on the first-order approximation in spin. This approximation is suitable for many cases because the ratio s/m is small, and spin orders higher than one can be neglected. At first order, the MPD equations simplify remarkably, such that velocity and momentum become aligned, and we can assume the spin magnitude is a conserved quantity.

In summary, in this work, we obtain an analytical solution for the MPD equations to first order in spin in a general plane gravitational wave spacetime with arbitrary polarization. Our approach is as follows: we construct a set of orthonormal tetrads that are parallel transported along the geodesic. We project the spin vector onto these tetrads and observe that the components of the spin vector in this basis behave like conserved quantities at first order in spin. This result is entirely due to the geometry of the problem, the symmetries, and our solution method based on moving to Rosen coordinates. In fact, because the metric for a general plane gravitational wave has been studied extensively, numerous symmetry properties have been discovered about it, some of which we will use to construct these conserved quantities.

Furthermore, we need three other conserved quantities to solve the MPD equations. Calculating these quantities is where the role of spin-gravity coupling emerges.

All the solutions within this work do not rely on a specific choice of wave profile or particular initial conditions; consequently, they are applicable to all plane wave profiles admissible in Rosen coordinates. Throughout this work, we use the metric signature is $(- + + +)$ and the natural system of units $G = c = 1$.

2 Introduction

We can describe the motion of a classical spinning particle using the Mathisson–Papapetrou–Dixon (MPD) equations [4] [26] [7] [17]

$$\begin{aligned}\frac{DP^\mu}{d\tau} &= -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}u^\nu S^{\alpha\beta} \\ \frac{DS^{\mu\nu}}{d\tau} &= P^\mu u^\nu - P^\nu u^\mu\end{aligned}$$

where u^μ is $\frac{dx^\mu}{d\tau}$, τ is an affine parameter, $R^\mu{}_{\nu\alpha\beta}$ is the Riemann tensor, $S^{\mu\nu}$ is the spin tensor, and P^μ is the four-momentum.

It is necessary to impose a Spin Supplementary Condition (SSC) [16] to determine the solution of the MPD equations uniquely.

We use the Tulczyjew–Dixon (TD) SSC [16] to specify the worldline of our classical spinning particle [7] [33]

$$S^{\mu\nu}P_\nu = 0$$

Additionally, we know that $S^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}P_\alpha s_\beta$ [20]. Where $\epsilon^{\mu\nu\alpha\beta} = \frac{\varepsilon^{\mu\nu\alpha\beta}}{\sqrt{-g}}$, $\varepsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita symbol, and g is the determinant of the metric. Generally, P^μ and u^μ do not necessarily coincide.

Throughout our notes, we consider only the linearized equations (linear in spin), so we can write [33] [3] [21]

$$P^\mu = mu^\mu + O(s^2) \quad (*)$$

It is clear that, at linear order in the spin, P^μ and u^μ coincide. In addition, the MPD equations reduce to [3] [21]

$$\begin{aligned}\frac{Du^\mu}{d\tau} &= -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}u^\nu \epsilon^{\alpha\beta\delta\lambda}u_\delta s_\lambda + O(s^2) \\ \frac{DS^{\mu\nu}}{d\tau} &= 0 + O(s^2)\end{aligned}$$

Using the last equation, and also $S^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}P_\alpha s_\beta$ [20], we clearly observe consistency with the Fermi–Walker transport of the spin vector. ($\frac{Ds^\mu}{d\tau} = \dot{u}^\nu u^\mu s_\nu$) [28] [10]

3 Conserved Quantities for the Plane Gravitational Wave Space-time

We study the interaction of a classical particle’s spin with a plane gravitational wave propagating in the x -direction, whose metric is given by

$$dS^2 = -dt^2 + dx^2 + (1 - h_+)dy^2 + (1 + h_+)dz^2 - 2h_\times dydz$$

We are interested in standard Rosen coordinates [8], so we can rewrite the metric as follows

$$dS^2 = 2dUdV + (1 - h_+)dy^2 + (1 + h_+)dz^2 - 2h_\times dydz \quad [5]$$

where

$$U := \frac{t - x}{\sqrt{2}} \quad V := \frac{-(t + x)}{\sqrt{2}}$$

On the other hand, the polarization of a gravitational wave has the following two forms [14]

$$\begin{aligned} h_+(U) &: + \text{ polarization modes of the gravitational wave [13][20]} \\ h_\times(U) &: \times \text{ polarization modes of the gravitational wave [13][20]} \end{aligned}$$

and U , in a sense, represents the retarded time, and V is advanced time. We can construct the metric matrix and compute its determinant as follows

$$g = \det \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 - h_+ & -h_\times \\ 0 & 0 & -h_\times & 1 + h_+ \end{bmatrix}$$

$$g = -1 + h_+^2 + h_\times^2$$

It is clear that, given the form of the metric, adding a constant to the V , y , or z directions does not change the metric. This indicates a symmetry along these directions, and we can define three of the Killing vectors as follows

$$\begin{aligned} k_{(V)}^\mu &= (\partial_V)^\mu & k_{(y)}^\mu &= (\partial_y)^\mu & k_{(z)}^\mu &= (\partial_z)^\mu \\ k_{(V)}^\mu &= [0 \ 1 \ 0 \ 0] & k_{(y)}^\mu &= [0 \ 0 \ 1 \ 0] & k_{(z)}^\mu &= [0 \ 0 \ 0 \ 1] \end{aligned}$$

The symmetries and Killing vectors in the background metric allow us to define a conserved quantity. (For details of the proof, see Appendix A.) This quantity is defined as follows [7] [29] [30]

$$J_{\vec{k}} = k_\mu P^\mu - \frac{1}{2} S^{\mu\nu} \nabla_\nu k_\mu$$

Now, let us compute this quantity for each of the three directions U, y , and z . The conserved quantity associated with the V direction is given by

$$\begin{aligned} \varepsilon := J_V &= g_{\mu\nu} k_{(V)}^\mu P^\alpha - \frac{1}{2} S^{\mu\nu} \nabla_\nu (g_{\mu\alpha} k_{(V)}^\alpha) = g_{01} k_{(V)}^1 P^0 - \frac{1}{2} S^{\mu\nu} \nabla_\nu (g_{\mu 1} k_{(V)}^1) = P^0 - \frac{1}{2} S^{0\nu} \underbrace{\nabla_\nu (1)}_{=0} \\ \varepsilon &= P^U \end{aligned} \quad (0)$$

which, in a sense, represents the longitudinal energy along the U direction. The conserved quantity associated with the y direction is given by:

$$J_{k_y} := J_I = g_{\mu\alpha} k_{(y)}^\alpha P^\mu - \frac{1}{2} S^{\mu\nu} \nabla_\nu (g_{\mu\alpha} k_{(y)}^\alpha)$$

Now, using the definition of the covariant derivative, we expand the above expression

$$J_I = g_{\mu\alpha} k_{(y)}^\alpha P^\mu - \frac{1}{2} S^{\mu\nu} (\partial_\nu k_{(y)\mu} - \Gamma_{\nu\mu}^\alpha k_{(y)\alpha}) = g_{\mu 2} P^\mu - \frac{1}{2} S^{\mu\nu} \partial_\nu k_{(y)\mu} + \underbrace{\frac{1}{2} S^{\mu\nu} \Gamma_{\nu\mu}^\alpha k_{(y)\alpha}}_{=0}$$

Terms like the last one, which result from the product of a symmetric part and an antisymmetric part, vanish. Now, by substituting the metric, we obtain

$$J_I = (1 - h_+) P^y - h_\times P^z - \frac{1}{2} S^{y\nu} \partial_\nu (1 - h_+) - \frac{1}{2} S^{z\nu} \partial_\nu (-h_\times)$$

h_+ and h_\times are functions of U only, so only their derivatives with respect to U are nonzero. Therefore, we have

$$J_I = (1 - h_+) P^y - h_\times P^z + \frac{1}{2} S^{yU} \dot{h}_+ + \frac{1}{2} S^{zU} \dot{h}_\times \quad (1)$$

Where the dots denote derivatives with respect to U . Now, we write the spin tensor in terms of the spin four-vector and the four-momentum vector, and substitute it into the above expression. On the other hand, we know that

$$S^{\mu\nu} = \frac{\varepsilon^{\mu\nu\alpha\beta}}{\sqrt{-g}} P_\alpha S_\beta \quad [20]$$

Using the above formula, we compute the following components of the spin tensor

$$\begin{aligned}
S^{20} &= \frac{\varepsilon^{2013}}{\sqrt{-g}}(P_1 s_3 - P_3 s_1) = \frac{1}{\sqrt{-g}} g_{1\mu} g_{3\nu} (P^\mu s^\nu - P^\nu s^\mu) \\
S^{20} &= \frac{g_{1\mu} g_{3\nu}}{\sqrt{-g}} P^{[\mu s^\nu]} = \frac{g_{3\nu}}{\sqrt{-g}} P^{[U s^\nu]} \\
S^{yU} &= \frac{1}{\sqrt{-g}} \left[(1 + h_+) P^{[U s^z]} - h_\times P^{[U s^y]} \right] \\
S^{30} &= \frac{\varepsilon^{3012}}{\sqrt{-g}}(P_1 s_2 - P_2 s_1) = \frac{-1}{\sqrt{-g}} g_{1\mu} g_{2\nu} (P^\mu s^\nu - P^\nu s^\mu) \\
S^{zU} &= \frac{g_{1\mu} g_{2\nu}}{\sqrt{-g}} P^{[\nu s^\mu]} = \frac{g_{2\nu}}{\sqrt{-g}} P^{[\nu s^U]} \\
S^{zU} &= \frac{1}{\sqrt{-g}} \left[(1 - h_+) P^{[y s^U]} - h_\times P^{[z s^U]} \right]
\end{aligned}$$

where $P^{[\mu s^\nu]}$ is equal to $P^\mu s^\nu - P^\nu s^\mu$.

Now, we substitute the above expressions into Equation (1)

$$\begin{aligned}
J_I &= (1 - h_+) P^y - h_\times P^z + \frac{1}{2} \frac{\dot{h}_+}{\sqrt{-g}} \left(-(1 + h_+) P^{[z s^U]} + h_\times P^{[y s^U]} \right) + \frac{1}{2} \frac{\dot{h}_\times}{\sqrt{-g}} \left((1 - h_+) P^{[y s^U]} - h_\times P^{[z s^U]} \right) \\
J_I &= (1 - h_+) P^y - h_\times P^z + \frac{P^{[y s^U]}}{2\sqrt{-g}} \left(\dot{h}_+ h_\times + \dot{h}_\times (1 - h_+) \right) - \frac{P^{[z s^U]}}{2\sqrt{-g}} \left(\dot{h}_+ (1 + h_+) + \dot{h}_\times h_\times \right) \quad (2)
\end{aligned}$$

Now, in a similar manner, we compute the conserved quantity associated with the z direction

$$\begin{aligned}
J_{II} &= g_{\mu\alpha} k_{(z)}^\alpha P^\mu - \frac{1}{2} S^{\mu\nu} (\partial_\nu k_{(z)\mu} - \Gamma_{\nu\mu}^\alpha k_{(z)\alpha}) = g_{\mu 3} P^\mu - \frac{1}{2} S^{\mu\nu} \partial_\nu g_{3\mu} + \underbrace{\frac{1}{2} S^{\mu\nu} \Gamma_{\nu\mu}^\alpha k_{(z)\alpha}}_{=0} \\
J_{II} &= -h_\times P^y + (1 + h_+) P^z - \frac{1}{2} S^{3\nu} \partial_\nu (1 + h_+) - \frac{1}{2} S^{2\nu} \partial_\nu (-h_\times) \frac{h_+ = h_+(U)}{h_\times = h_\times(U)} \\
J_{II} &= (1 + h_+) P^z - h_\times P^y + \frac{\dot{h}_\times}{2\sqrt{-g}} \left((1 + h_+) P^{[U s^z]} - h_\times P^{[U s^y]} \right) - \frac{\dot{h}_+}{2\sqrt{-g}} \left((1 - h_+) P^{[y s^U]} - h_\times P^{[z s^U]} \right) \\
J_{II} &= (1 + h_+) P^z - h_\times P^y + \frac{P^{[z s^U]}}{2\sqrt{-g}} \left(\dot{h}_\times (1 + h_+) - \dot{h}_+ h_\times P^{[U s^y]} \right) - \frac{P^{[y s^U]}}{2\sqrt{-g}} \left(\dot{h}_+ (1 - h_+) - \dot{h}_\times h_\times P^{[U s^y]} \right) \quad (3)
\end{aligned}$$

J_I and J_{II} represent, in a sense, the conserved angular momenta along the y and z directions, respectively. In the absence of spin, the above equations take the following form

$$\begin{cases} J_I^{(0)} = (1 - h_+) P_{(0)}^y - h_\times P_{(0)}^z \\ J_{II}^{(0)} = (1 + h_+) P_{(0)}^z - h_\times P_{(0)}^y \end{cases}$$

where the index (0) denotes the order of $O(s^0)$. From the above equations, we can easily write the four-momenta to order $O(s^0)$ as follows

$$P_{(0)}^y = \frac{(1 + h_+) J_I^{(0)} + h_\times J_{II}^{(0)}}{-g} \quad (4)$$

$$P_{(0)}^z = \frac{h_\times J_I^{(0)} + (1 - h_+) J_{II}^{(0)}}{-g} \quad (5)$$

4 Parallel-Transported Tetrad Method

The conserved quantities ε , J_I , and J_{II} , which we obtained from the Killing symmetries of the plane gravitational wave metric, are not sufficient to determine all the components of the momentum in the MPD equations. In fact, unlike the spinless case, here the spin–gravity coupling introduces additional degrees of freedom through the spin four-vector, and it is necessary to find other conserved quantities (still to first order in spin) in order to fully solve the MPD equations. Next, we will use the method developed by Skoupý, Witzany [32], and Marck [18] to construct a set of parallel-transported orthonormal tetrads, and we will see that expressing the spin in this basis adds three additional conserved quantities. We take the zeroth leg of the tetrad to be the four-velocity vector

$$e_0^\mu = u^\mu \quad (6)$$

Since $u^\mu u_\mu = -1$, this tetrad is normalized.

One of the properties of a plane gravitational wave in Rosen coordinates is the existence of a covariantly constant null vector field $l^\mu = (\partial_V)^\mu$ that satisfies the $\nabla_\alpha l^\mu = 0$ condition [1]. Consequently, we can define the next tetrad vector as follows

$$e_1^\mu = \frac{1}{u_0} (\partial_V)^\mu + u^\mu \quad (7)$$

which is clearly orthogonal to the previous tetrad and has unit norm. We then introduce the other two “primary” tetrads in the transverse plane as follows

$$\tilde{e}_2^\mu = \frac{(\partial_y)^\mu}{\sqrt{1-h_+}} + \frac{u^3 h_\times - u^2(1-h_+)}{u^0 \sqrt{1-h_+}} (\partial_V)^\mu \quad (8)$$

$$\tilde{e}_3^\mu = \sqrt{\frac{1-h_+}{-g}} \left(\frac{h_\times (\partial_y)^\mu}{1-h_+} + (\partial_z)^\mu + \frac{g u^3}{u^0 (1-h_+)} (\partial_V)^\mu \right) \quad (9)$$

these two tetrads, together with the previous two, form an orthonormal basis. (For the full details of the proof, see Appendix B.)

Now, let us consider the spinless case. In this case, the four-velocity is tangent to the geodesic, and we denote this four-vector by u_g^μ . Clearly, in this situation, the first two tetrads we obtained are parallel transported, but the last two tetrads are not. Since the transverse plane is two-dimensional, we can rotate the last two tetrads by an angle ψ so that they become parallel transported. We define them as follows

$$\left\{ \begin{array}{l} e_2^\mu = \tilde{e}_2^\mu \cos \psi_{(U)} - \tilde{e}_3^\mu \sin \psi_{(U)} \\ e_3^\mu = \tilde{e}_2^\mu \sin \psi_{(U)} + \tilde{e}_3^\mu \cos \psi_{(U)} \end{array} \right\}$$

We then require that these two tetrads be parallel transported along u_g , which leads to

$$\frac{d\psi}{dU} = \frac{-1}{2\sqrt{-g}} \left[\frac{h_\times \dot{h}_+}{1-h_+} + \dot{h}_\times \right] \quad (10)$$

Based on the above equation, the full evolution of the rotation angle can be determined given the initial conditions. (For the full details of the proof, see Appendix C.)

Now, to take advantage of the structure we built with the tetrads, let us decompose the spin vector as follows

$$s^\mu = s_I e_1^\mu + s_{II} e_2^\mu + s_{III} e_3^\mu \quad (11)$$

We do not consider the component along U because, according to the formula below, the spin vector is orthogonal to it. In fact, we have

$$\begin{aligned} s^\mu &= -\frac{1}{2M^2} \epsilon^{\mu\nu\alpha\beta} P_\nu S_{\alpha\beta} \\ s^\mu u_\mu &= \frac{-1}{2M^2} \epsilon^{\mu\nu\alpha\beta} u_\mu P_\nu S_{\alpha\beta} \quad P_\nu = m u_\nu + O(s^2) \\ s^\mu u_\mu &\approx \frac{m}{2M^2} \underbrace{\epsilon^{\mu\nu\alpha\beta}}_{\text{anti-symmetric in } \mu, \nu} \underbrace{u_\mu u_\nu}_{\text{symmetric in } \mu, \nu} S_{\alpha\beta} = 0 \\ s^\mu u_\mu &\approx 0 \end{aligned}$$

Now, using the MPD equations to first order in spin, we have

$$\frac{DS^{\mu\nu}}{d\tau} \approx 0 \quad \xrightarrow{\frac{D e_i^\mu}{d\tau} = 0} \quad \frac{ds_I}{d\tau} = \frac{ds_{II}}{d\tau} = \frac{ds_{III}}{d\tau} = 0 + O(s^2)$$

Which means s_I , s_{II} , and s_{III} are constants of motion. These three quantities, together with ε , J_I , and J_{II} are sufficient to provide a complete set of equations to fully solve the MPD equations.

5 Explicit Solution of the MPD Equations to First Order in Spin

In this section, we aim to explicitly solve the MPD equations to first order in spin, using the Killing constants as well as the spin constants we have obtained. First, using Equations (6), (7), (8), (9), and (11), we compute the components of the spin vector. For the U component, we have

$$s^U = u^0 s_I \rightarrow s^U = \frac{P^0 s_I}{m} = \varepsilon \left(\frac{s_I}{m} \right) \rightarrow s^U = \varepsilon \left(\frac{s_I}{m} \right)$$

By replacing $\frac{s_I}{m}$ with s_I (we are only renaming here!), we obtain

$$s^U = \varepsilon s_I \quad (12)$$

Now, let us compute the y component of the spin vector in a similar manner, using Equations (*), (6), (7), (8), (9), and (11)

$$\begin{aligned} s^y &= s_I e_1^y + s_{II} e_2^y + s_{III} e_3^y \\ s^y &= s_I (u^y) + s_{II} \left(\frac{\cos \psi}{\sqrt{1-h_+}} - \sin \psi \frac{h_\times}{\sqrt{-g(1-h_+)}} \right) + s_{III} \left(\frac{\sin \psi}{\sqrt{1-h_+}} + \cos \psi \frac{h_\times}{\sqrt{-g(1-h_+)}} \right) \\ s^y &= P_{(0)}^y \left(\frac{s_I}{m} \right) + \frac{s_{II} \cos \psi + s_{III} \sin \psi}{\sqrt{1-h_+}} + \frac{s_{III} \cos \psi - s_{II} \sin \psi}{\sqrt{-g(1-h_+)}} h_\times \\ s^y &= \frac{(1+h_+)J_I^{(0)} + h_\times J_{II}^{(0)}}{-g} \left(\frac{s_I}{m} \right) + \frac{s_{II} \cos \psi + s_{III} \sin \psi}{\sqrt{1-h_+}} + \frac{s_{III} \cos \psi - s_{II} \sin \psi}{\sqrt{-g(1-h_+)}} h_\times \quad \xrightarrow{\left(\frac{s_I}{m}\right) \rightarrow s_I} \\ s^y &= \frac{(1+h_+)J_I + h_\times J_{II}}{-g} s_I + \frac{s_{II} \cos \psi + s_{III} \sin \psi}{\sqrt{1-h_+}} + \frac{s_{III} \cos \psi - s_{II} \sin \psi}{\sqrt{-g(1-h_+)}} h_\times \end{aligned} \quad (13)$$

Similarly, for the z component, we have

$$\begin{aligned} s^z &= s_I e_1^z + s_{II} e_2^z + s_{III} e_3^z \\ s^z &= s_I u^z + s_{II} \sqrt{\frac{1-h_+}{-g}} (-\sin \psi) + s_{III} \cos \psi \sqrt{\frac{1-h_+}{-g}} \quad \xrightarrow{\left(\frac{s_I}{m}\right) \rightarrow s_I} \\ s^z &= P_{(0)}^z s_I + \sqrt{\frac{1-h_+}{-g}} (-s_{II} \sin \psi + s_{III} \cos \psi) \\ s^z &= \frac{h_\times J_I + (1-h_+)J_{II}}{-g} s_I + \sqrt{\frac{1-h_+}{-g}} (-s_{II} \sin \psi + s_{III} \cos \psi) \end{aligned} \quad (14)$$

Using Equations (0), (2), and (3), as well as Equations (12), (13), and (14), one can form a system of linear equations and compute the four-momentum to first order in spin. We can write the four-momentum as the four-momentum in the spinless case plus first-order corrections in spin, as follows

$$P^y = \frac{J_I(1+h_+) + J_{II}(h_\times)}{-g} + F_U^{(y)} s^U + F_y^{(y)} s^y + F_z^{(y)} s^z \quad (15)$$

$$P^z = \frac{h_\times J_I + (1-h_+)J_{II}}{-g} + F_U^{(z)} s^U + F_y^{(z)} s^y + F_z^{(z)} s^z \quad (16)$$

where the functions F depend only on the retarded time U . An important point to note is that we have written the first term without the (0) indices. In fact, since we have the quantities J themselves — and

not merely their zeroth-order values in spin — we write the quantity itself in the first term instead of explicitly indicating the zeroth order.

We assume that the first-order spin contributions in the first term are effectively absorbed through a suitable redefinition of the coefficients F , such that the total expression remains consistent with the previous equations. In other words, this is nothing but a redefinition of the coefficients F . We are free to appropriately redefine both the coefficients F and the first term so that the sum of the correction terms and the leading term remains unchanged and consistent with the earlier equations.

Moreover, according to the assumption stated at the beginning, we neglect terms of order higher than second in spin. By substituting Equations (15) and (16) into Equations (2) and (3), and equating the coefficients of the independent components of the spin vector, the functions F can be determined. The results are as follows

$$F_z^{(y)} = \frac{-\varepsilon[(1+h_+)C_2 + C_3h_\times]}{-g} \quad (17)$$

$$F_z^{(z)} = \frac{-\varepsilon(C_2h_\times + C_3(1-h_+))}{-g} \quad (18)$$

$$F_y^{(y)} = \frac{\varepsilon(C_1(h_\times^2 - g) - C_4h_\times(1-h_+))}{-g(1-h_+)} \quad (19)$$

$$F_y^{(z)} = \frac{\varepsilon(C_1h_\times - C_4(1-h_+))}{-g} \quad (20)$$

$$F_U^{(y)} = \frac{C_2(h_\times^2 - g)Q^z + (h_\times(1-h_+)C_4 - C_1(h_\times^2 - g))Q^y}{-g(1-h_+)} \quad (21)$$

$$F_U^{(z)} = \frac{C_2Q^zh_\times + [-h_\times C_1 + (1-h_+)C_4]Q^y}{-g} \quad (22)$$

where

$$C_1 := \frac{\dot{h}_+h_\times + \dot{h}_\times(1-h_+)}{2\sqrt{-g}} \quad C_2 := \frac{\dot{h}_+(1+h_+) + \dot{h}_\times h_\times}{2\sqrt{-g}} \quad C_3 := \frac{\dot{h}_\times(1+h_+) - \dot{h}_+h_\times}{2\sqrt{-g}}$$

$$C_4 := \frac{\dot{h}_+(1-h_+) - \dot{h}_\times h_\times}{2\sqrt{-g}} \quad Q^y := \frac{J_I(1+h_+) + J_{II}(h_\times)}{-g} \quad Q^z := \frac{h_\times J_I + (1-h_+)J_{II}}{-g}$$

For the full details of the derivation of the above equations, see Appendix D.

According to Equations (*) and (0), we have

$$\frac{dU}{d\tau} = u^U = \frac{P^U}{m} = \frac{\varepsilon}{m} \quad (**)$$

The retarded time, U , is similar to an affine parameter along the particle's worldline. Using the chain rule, we have

$$\left\{ \begin{array}{l} \frac{dy}{dU} = \frac{u^y}{u^U} = \frac{P_{(U)}^y}{\varepsilon} \\ \frac{dz}{dU} = \frac{u^z}{u^U} = \frac{P^z}{P^U} = \frac{P_{(U)}^z}{\varepsilon} \end{array} \right\}$$

By integrating the above relations, we arrive at the general form of the solution for the transverse coordinates

$$y_{(U)} = y_{(0)} + \frac{1}{\varepsilon} \int_0^U P_{(U')}^y dU' \quad (23)$$

$$Z_{(U)} = z_{(0)} + \frac{1}{\varepsilon} \int_0^U P_{(U')}^z dU' \quad (24)$$

By dividing the background metric equation by dU^2 , we have

$$\frac{dS^2}{dU^2} = 2\frac{dV}{dU} + (1-h_+) \left(\frac{dy}{dU} \right)^2 + (1+h_+) \left(\frac{dz}{dU} \right)^2 - 2h_\times \frac{dy}{dU} \frac{dz}{dU}$$

Now, from Equation (**), we obtain

$$-\frac{1}{\left(\frac{pU}{m}\right)^2} = -\left(\frac{m}{\varepsilon}\right)^2 = 2\frac{dV}{dU} + (1 - h_+) \left(\frac{dy}{dU}\right)^2 + (1 + h_+) \left(\frac{dz}{dU}\right)^2 - 2h_\times \frac{dy}{dU} \frac{dz}{dU}$$

Using $x = \frac{-U-V}{\sqrt{2}}$, we obtain

$$\frac{dx}{dU} = \frac{-1}{\sqrt{2}} \left(1 + \frac{dV}{dU}\right)$$

By substituting the above expression into the metric equation, we have

$$-\left(\frac{m}{\varepsilon}\right)^2 = 2 \left[-\sqrt{2} \frac{dx}{dU} - 1 \right] + (1 - h_+) \left(\frac{dy}{dU}\right)^2 + (1 + h_+) \left(\frac{dz}{dU}\right)^2 - 2h_\times \frac{dy}{dU} \frac{dz}{dU}$$

Finally, by integrating the above expression over the retarded time, we obtain

$$x_{(U)} = x_{(0)} - \frac{U}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \int_0^U dU' \left\{ (1 - h_+) \left(\frac{dy}{dU'}\right)^2 + (1 + h_+) \left(\frac{dz}{dU'}\right)^2 - 2h_\times \frac{dy}{dU'} \frac{dz}{dU'} + \left(\frac{m}{\varepsilon}\right)^2 \right\} \quad (25)$$

To fully determine the worldline of a particle in the presence of a plane gravitational wave, the six conserved quantities we previously established ($\varepsilon, J_I, J_{II}, s_I, s_{II}, s_{III}$), must be fixed using the initial conditions at $U = 0$. According to Equations (0), (2), and (3), specifying the initial position four-vector, the initial four-momentum at $U = 0$, and the initial spin vector is sufficient to determine these conserved quantities. (Clearly, since these quantities are conserved, their values remain constant at all times; therefore, once we compute them at $U = 0$ based on the initial physical quantities such as position, four-momentum, and spin vector, their values at any other time are automatically fixed.)

The only gauge freedom that remains is $\psi_{(U)}$.

6 Discussion

The spin evolution equations are given by equations 12, 13, and 14. Let us examine this set of equations for a particle that is initially at rest, in the case where the particle's spin is aligned with the direction of wave propagation (the x-direction). We know that the six conserved quantities we have calculated remain constant at all times. Therefore, to determine their values, it is sufficient to evaluate them at the initial time ($U = 0$). Since we have assumed that the particle is initially at rest, it follows from Equations (2) and (3) that J_I and J_{II} are equal to zero. Given that these quantities vanish, $F_U^{(y)}$ and $F_U^{(z)}$ are also zero, which means that the s^U component does not contribute to the momentum. On the other hand, since we have chosen the spin to be aligned with the direction of wave propagation, (11), together with the orthogonality of the tetrad basis we constructed, implies that the components s_{II} and s_{III} are zero. This means that all first-order spin corrections in the equations vanish, the momentum reduces exactly to the spinless case, and the particle's trajectory becomes the same as the geodesic.

This implies that, to first order in spin, the gravitational wave does not couple to a particle whose spin is aligned with the direction of wave propagation.

Another case that we are interested in examining is the situation in which the particle's spin is perpendicular to the direction of propagation of the plane gravitational wave and has no component along the propagation direction.

Using equations 12, 13, and 14, which describe the spin evolution, and assuming that the polarization of the gravitational wave is given by $h_+ = h \sin(\eta U)$ and $h_\times = h \sin(\eta U + \delta)$, Figure 1 can be plotted for different values of δ to illustrate the spin evolution.

The evolution of the spin vector has previously been studied within various frameworks of the standard theory of gravity. An important point is that, given the particular way we set the polarization of the gravitational wave, the plots drawn in Figure 1 are very similar to the spin-evolution plots of a gyroscope in the Earth's gravitational field in the linear-gravity regime. In fact, the spin precession predicted by Lense–Thirring precession [27] [20], particularly when viewed in the framework of Gravitoelectromagnetism [19] [13] [20] [6] [12], is very similar to the precessional motion that we have derived here using the

MPD equations, although for the plane gravitational wave metric. Another important point is that for a gyroscope in Earth orbit the precessional motion is a combination of two types: one is the Lense–Thirring precession and the other is the de Sitter precession [9] [31], which arises from the parallel transport of the gyroscope’s spin vector. The de Sitter precession was found to be fully consistent with the results of the Gravity Probe B experiment [11] [15]. This parallel transport is exactly what we initially obtained as $\frac{Ds^\mu}{d\tau} = 0$, the linearized form of one of the MPD equations, and it is fully consistent with Fermi–Walker transport. As a result, it appears that the spin evolution equation in the linear regime is the same for the de Sitter precession and plane gravitational wave ($\frac{Ds^\mu}{d\tau} = 0$). Therefore, the similarity between the graphs was not unexpected. The differences observed between the graphs arise from the differences in the metric governing each of the phenomena. Ultimately, studying the MPD equations can provide a framework for looking at the spin-evolution problem from another approach and for comparing the consistency of different methods in describing spin evolution in gravity.

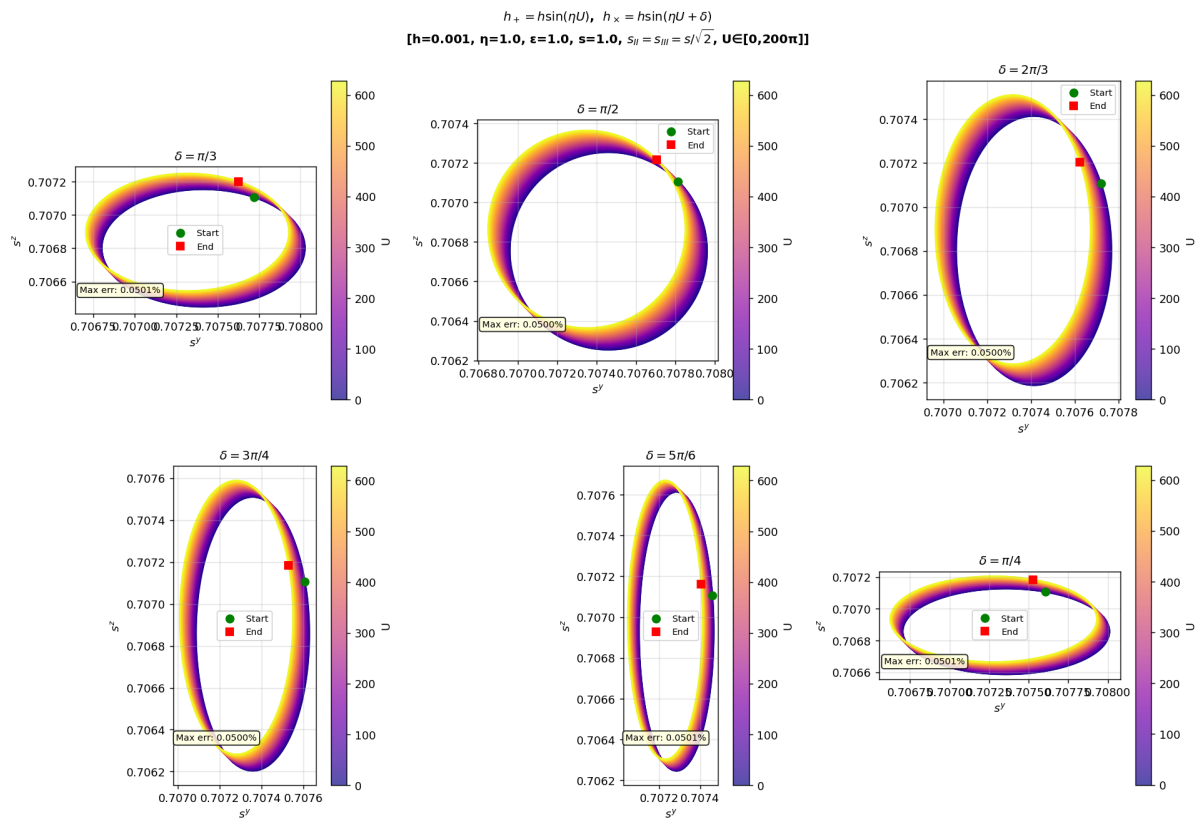


Figure 1: Spin Evolution

7 Appendix A

We start with the Killing equation and then take the covariant derivative of it.

$$\begin{aligned}\nabla_\alpha k_\beta + \nabla_\beta k_\alpha &= 0 \quad [2] \\ \rightarrow \nabla_\nu \nabla_\alpha k_\beta + \nabla_\nu \nabla_\beta k_\alpha &= 0 \\ \nabla_\nu \nabla_\alpha k_\beta + [\nabla_\nu, \nabla_\beta] k_\alpha + \nabla_\beta \nabla_\nu k_\alpha &= 0\end{aligned}$$

where k is Killing vector.

According to the definition of the Riemann tensor based on the commutator of covariant derivatives [34], we have

$$\begin{aligned}\nabla_\nu \nabla_\alpha k_\beta &= R^\gamma{}_{\alpha\nu\beta} k_\gamma - \nabla_\beta \nabla_\nu k_\alpha \\ \nabla_\nu k_\alpha + \nabla_\alpha k_\nu = 0 \rightarrow \nabla_\beta \underbrace{\nabla_\nu k_\alpha}_{-\nabla_\alpha k_\nu} + \nabla_\beta \nabla_\alpha k_\nu &= 0 \xrightarrow{(*)} \nabla_\nu \nabla_\alpha k_\beta = R^\gamma{}_{\alpha\nu\beta} k_\gamma + \nabla_\beta \nabla_\alpha k_\nu \rightarrow \\ \nabla_\nu \nabla_\alpha k_\beta &= R^\gamma{}_{\alpha\nu\beta} k_\gamma + \underbrace{[\nabla_\beta, \nabla_\alpha]}_{-R^\gamma{}_{\nu\beta\alpha} k_\gamma} k_\nu + \nabla_\alpha \nabla_\beta k_\nu = \nabla_\alpha \nabla_\beta k_\nu + R_{\nu\gamma\beta\alpha} k^\gamma - R_{\alpha\gamma\nu\beta} k^\gamma = -\nabla_\alpha \nabla_\nu k_\beta + R_{\nu\gamma\beta\alpha} k^\gamma - R_{\alpha\gamma\nu\beta} k^\gamma \\ \begin{cases} \nabla_\alpha \nabla_\nu k_\beta + \nabla_\nu \nabla_\alpha k_\beta = (R_{\nu\gamma\beta\alpha} - R_{\alpha\gamma\nu\beta}) k^\gamma & \rightarrow \\ [\nabla_\alpha, \nabla_\nu] k_\beta = -R_{\gamma\beta\alpha\nu} k^\gamma \end{cases} \\ 2\nabla_\alpha \nabla_\nu k_\beta &= (R_{\nu\gamma\beta\alpha} - R_{\alpha\gamma\nu\beta} - R_{\gamma\beta\alpha\nu}) k^\gamma \\ \begin{cases} 2\nabla_\alpha \nabla_\nu k_\beta = (-R_{\gamma\nu\beta\alpha} + R_{\gamma\alpha\nu\beta} - R_{\gamma\beta\alpha\nu}) k^\gamma & \rightarrow \\ R_{\gamma\nu\beta\alpha} + R_{\gamma\alpha\nu\beta} + R_{\gamma\beta\alpha\nu} = 0 \end{cases} \\ \nabla_\alpha \nabla_\nu k_\beta &= R_{\gamma\alpha\nu\beta} k^\gamma \\ \nabla_\alpha \nabla_\nu k_\beta &= R^\gamma{}_{\alpha\nu\beta} k_\gamma\end{aligned}\tag{A1}$$

Now, according to the identity we proved, we demonstrate that J is a conserved quantity

$$\begin{aligned}J_{\vec{k}} &:= k_\mu P^\mu - \frac{1}{2} S^{\mu\nu} \nabla_\nu k_\mu \\ \frac{dJ_{\vec{k}}}{d\tau} &= u^\alpha \nabla_\alpha (k_\mu P^\mu) - \frac{1}{2} u^\alpha \nabla_\alpha (S^{\mu\nu} \nabla_\nu k_\mu) = u^\alpha (\nabla_\alpha k_\mu) P^\mu + k_\mu \frac{DP^\mu}{d\tau} - \frac{1}{2} u^\alpha (\nabla_\alpha S^{\mu\nu} \nabla_\nu k_\mu + S^{\mu\nu} \nabla_\alpha \nabla_\nu k_\mu) \\ \frac{dJ_{\vec{k}}}{d\tau} &= P^\mu u^\alpha \nabla_\alpha k_\mu + k_\mu \left(-\frac{1}{2} R^\mu{}_{\nu\alpha\beta} u^\nu S^{\alpha\beta} \right) - \frac{1}{2} u^\alpha (\nabla_\alpha S^{\mu\nu} \nabla_\nu k_\mu + S^{\mu\nu} \nabla_\alpha \nabla_\nu k_\mu)\end{aligned}$$

Using Equation (A1), we have

$$\begin{aligned}\frac{dJ_{\vec{k}}}{d\tau} &= P^\mu u^\alpha \nabla_\alpha k_\mu - \frac{1}{2} R^\mu{}_{\nu\alpha\beta} k_\mu u^\nu S^{\alpha\beta} - \frac{1}{2} u^\alpha \nabla_\alpha S^{\mu\nu} \nabla_\nu k_\mu - \frac{1}{2} u^\alpha S^{\mu\nu} R^\mu{}_{\alpha\nu\mu} k_\mu \\ \frac{dJ_{\vec{k}}}{d\tau} &= P^\mu u^\alpha \nabla_\alpha k_\mu - \frac{1}{2} u^\alpha (\nabla_\alpha S^{\mu\nu}) \nabla_\nu k_\mu = P^\mu u^\alpha \nabla_\alpha k_\mu - \frac{1}{2} (P^\mu u^\nu - P^\nu u^\mu) \nabla_\nu k_\mu = \frac{1}{2} P^\mu u^\nu \nabla_\nu k_\mu + \frac{1}{2} P^\nu u^\mu \nabla_\nu k_\mu \\ \frac{dJ_{\vec{k}}}{d\tau} &= \frac{1}{2} (P^\mu u^\nu + P^\nu u^\mu) \nabla_\nu k_\mu \stackrel{\mu \leftrightarrow \nu}{=} \frac{1}{2} (P^\nu u^\mu + P^\mu u^\nu) \nabla_\mu k_\nu = -\frac{1}{2} (P^\mu u^\nu + P^\nu u^\mu) \nabla_\nu k_\mu \Rightarrow \\ \frac{dJ_{\vec{k}}}{d\tau} &= \frac{1}{2} (P^\mu u^\nu + P^\nu u^\mu) \nabla_\nu k_\mu = 0 \\ \frac{dJ_{\vec{k}}}{d\tau} &= 0\end{aligned}$$

8 Appendix B

What we are going to do here to construct orthonormal bases is very similar to the Gram–Schmidt process. We have four vectors in the directions $U, V, y,$ and z . We want to construct four orthonormal vectors from the linear combination of these vectors. Throughout the Calculations, we make use of relation $e^\mu_{\hat{a}} e^\nu_{\hat{b}} g_{\mu\nu} = \eta_{\hat{a}\hat{b}}$ [25].

$$\begin{aligned} e_0^\mu &:= u^\mu \\ l^\mu &: \text{It is a covariantly constant null vector field } l^\mu l_\mu = 0 \quad \nabla_\alpha l^\mu = 0 \\ l^\mu &= (\partial_V)^\mu \\ l^\mu &= (0 \ 1 \ 0 \ 0) \end{aligned}$$

Now, we consider the second vector, which is along the V direction, and add it to the set and construct a tetrad from a linear combination of l^μ and u^μ . This tetrad must have unit norm and be orthogonal to the previous tetrad. By imposing these two conditions, we determine the unknown coefficients.

$$\begin{aligned} e_1^\mu &= Al^\mu + Bu^\mu \\ u_\mu e_1^\mu = 0 &\rightarrow Al^\mu u_\mu - B = 0 \rightarrow B = Al^\mu u_\mu \rightarrow e_1^\mu = A(l^\mu + l^\alpha u_\alpha u^\mu) \\ e_1^\mu : \text{normalization} &\rightarrow A^2(l^\mu + l^\alpha u_\alpha u^\mu)(l_\mu + l^\alpha u_\alpha u_\mu) = 1 = A^2 \underbrace{(l^\mu l_\mu + (l^\alpha u_\alpha)^2 + (l^\alpha u_\alpha)^2 - (l^\alpha u_\alpha)^2)}_0 \\ &\rightarrow 1 = A^2 (l^\alpha u_\alpha)^2 \xrightarrow{\text{if } A > 0} \boxed{A = \frac{1}{l^\alpha u_\alpha}} \\ l^\alpha u_\alpha &= l^\alpha u^\beta g_{\alpha\beta} = l^1 u^\beta g_{1\beta} = u^\beta g_{1\beta} = u^0 \rightarrow \boxed{A = \frac{1}{u^0}} \\ e_1^\mu &= \frac{1}{u^0} (l^\mu + u^0 u^\mu) \rightarrow \boxed{e_1^\mu = \frac{1}{u^0} (\partial_V)^\mu + u^\mu} \\ (\partial_y)^\mu &\xrightarrow{\text{normalization}} C^2 (\partial_y)^\mu (\partial_y)^\nu g_{\mu\nu} = 1 \\ C^2 (\partial_y)^2 (\partial_y)^2 g_{22} = 1 &\rightarrow C^2 (1 - h_+) = 1 \xrightarrow{\text{if } C > 0} C = \frac{1}{\sqrt{1 - h_+}} \Rightarrow (\partial_y)^\mu \rightarrow \frac{(\partial_y)^\mu}{\sqrt{1 - h_+}} \\ (\partial_z)^\mu &\xrightarrow{\text{normalization}} D^2 (\partial_z)^\mu (\partial_z)^\nu g_{\mu\nu} = 1 \\ D^2 (\partial_z)^2 (\partial_z)^2 g_{22} = 1 &\rightarrow D^2 (1 + h_+) = 1 \xrightarrow{\text{if } D > 0} D = \frac{1}{\sqrt{1 + h_+}} \Rightarrow (\partial_z)^\mu \rightarrow \frac{(\partial_z)^\mu}{\sqrt{1 + h_+}} \end{aligned}$$

Now, we add the third vector, which lies along the y direction, to the set. By again imposing the two conditions — unit normalization and orthogonality of the new tetrad to the previous two — we determine the coefficients. It is clear that since this tetrad must be orthogonal to the first tetrad, which lies along u , and since $u^\mu u_\mu = -1$, it must not contain any term proportional to u . In other words, the coefficient in front of u in this tetrad (and also in the next tetrad we will construct) is zero.

$$\begin{aligned} \tilde{e}_2^\mu &= A(\partial_V)^\mu + B \frac{(\partial_y)^\mu}{\sqrt{1 - h_+}} \quad \tilde{e}_2^\mu u_\mu = 0 \rightarrow A \underbrace{l^\mu u_\mu}_{u^0} + B \frac{(\partial_y)^\mu}{\sqrt{1 - h_+}} u_\mu = 0 \\ Au^0 + \frac{B}{\sqrt{1 - h_+}} [(\partial_y)^2 u^\nu g_{2\nu}] &= 0 \\ Au^0 + \frac{B}{\sqrt{1 - h_+}} [u^2 g_{22} + u^3 g_{23}] &= Au^0 + \frac{B}{\sqrt{1 + h_+}} [u^2 (1 - h_+) + u^3 (-h_\times)] = 0 \\ \rightarrow A &= \frac{B}{u^0 \sqrt{1 - h_+}} [u^3 h_\times - u^2 (1 - h_+)] \end{aligned}$$

$$\begin{aligned}
\tilde{e}_2^\mu &= \frac{B}{\sqrt{1-h_+}} \left[\underbrace{\frac{u^3 h_\times - u^2(1-h_+)}{u^0}}_{:=\chi} (\partial_V)^\mu + (\partial_y)^\mu \right] \\
&\xrightarrow{\text{normalization}} \frac{B^2}{1-h_+} [\chi (\partial_V)^\mu + (\partial_y)^\mu] [\chi (\partial_V)_\mu + (\partial_y)_\mu] = \\
&\frac{B^2}{1-h_+} [\chi^2 (\partial_V)^1 (\partial_V)^1 \underbrace{g_{11}}_{=0} + 2\chi (\partial_V)^1 (\partial_y)^2 \underbrace{g_{12}}_{=0} + (\partial_y)^2 (\partial_y)^2 \underbrace{g_{22}}_{=1-h_+}] = 1 \\
&\xrightarrow{\text{if } B>0} B = 1 \rightarrow \boxed{\tilde{e}_2^\mu = \frac{(\partial_y)^\mu}{\sqrt{1-h_+}} + \frac{u^3 h_\times - u^2(1-h_+)}{u^0 \sqrt{1-h_+}} (\partial_V)^\mu}
\end{aligned}$$

Again, we proceed in a similar manner. This time, we add the vector along the z direction and impose orthogonality to the other tetrads as well as the unit normalization condition.

$$\begin{aligned}
\tilde{e}_3^\mu &= A(\partial_y)^\mu + B(\partial_z)^\mu + C(\partial_V)^\mu \\
\tilde{e}_2^\mu &= \alpha(\partial_y)^\mu + \beta(\partial_V)^\mu
\end{aligned}$$

From the orthogonality of tetrads 2 and 3, we have

$$\begin{aligned}
(A(\partial_y)^\mu + B(\partial_z)^\mu + C(\partial_V)^\mu)(\alpha(\partial_y)_\mu + \beta(\partial_V)_\mu) &= 0 \rightarrow A(1-h_+) - Bh_\times = 0 \rightarrow A = \frac{Bh_\times}{1-h_+} \\
\rightarrow \tilde{e}_3^\mu &= B \left(\frac{h_\times (\partial_y)^\mu}{1-h_+} + (\partial_z)^\mu \right) + C(\partial_V)^\mu \\
\tilde{e}_1^\mu &= \frac{(\partial_V)^\mu}{u^0} + u^\mu
\end{aligned}$$

From the orthogonality of tetrads 1 and 3, we have

$$\begin{aligned}
\rightarrow \left(B \left(\frac{h_\times (\partial_y)^\mu}{1-h_+} + (\partial_z)^\mu \right) + C(\partial_V)^\mu \right) \left(\frac{(\partial_V)_\mu}{u^0} + u_\mu \right) &= 0 \\
\frac{Bh_\times (\partial_y)^\mu}{1-h_+} u_\mu + B(\partial_z)^\mu u_\mu + C \underbrace{(\partial_V)^\mu u_\mu}_{u^0} &= 0 \\
C = -\frac{B}{u^0} \left(\frac{h_\times}{1-h_+} (u^2(1-h_+) - h_\times u^3) + u^3(1+h_+) - h_\times u^2 \right) &= C = -\frac{B}{u^0} \left(\frac{-h_\times^2}{1-h_+} + 1 + h_+ \right) u^3 \\
\boxed{C = -\frac{Bu^3}{u^0} \left(\frac{1-h_+^2 - h_\times^2}{1-h_+} \right) = \frac{gBu^3}{u^0(1-h_+)}} & \\
\tilde{e}_3^\mu = B \left(\frac{h_\times (\partial_y)^\mu}{1-h_+} + (\partial_z)^\mu + \frac{gu^3}{u^0(1-h_+)} (\partial_V)^\mu \right) &\xrightarrow{\text{normalization}} \\
1 = B^2 \left(\frac{h_\times (\partial_y)^\mu}{1-h_+} + (\partial_z)^\mu + \frac{gu^3}{u^0(1-h_+)} (\partial_V)^\mu \right) \left(\frac{h_\times (\partial_y)_\mu}{1-h_+} + (\partial_z)_\mu + \frac{gu^3}{u^0(1-h_+)} (\partial_V)_\mu \right) & \\
1 = B^2 \left(\frac{h_\times^2 (1-h_+)}{(1-h_+)^2} - \frac{h_\times^2}{(1-h_+)^2} - \frac{h_\times^2}{(1-h_+)^2} + (1-h_+) \right) &= B^2 \left(\frac{1-h_+^2 - h_\times^2}{1-h_+} \right) \\
1 = -\frac{B^2 g}{1-h_+} \xrightarrow{\text{if } B>0} \boxed{B = \sqrt{\frac{1-h_+}{-g}}} &\Rightarrow \\
\boxed{\tilde{e}_3^\mu = \sqrt{\frac{1-h_+}{-g}} \left(\frac{h_\times (\partial_y)^\mu}{1-h_+} + (\partial_z)^\mu + \frac{gu^3}{u^0(1-h_+)} (\partial_V)^\mu \right)} &
\end{aligned}$$

9 Appendix C

$$\begin{aligned}
& \text{let } u^\mu \text{ be Geodesic } \quad u^\mu = u_g^\mu \\
& \frac{Du_g^\mu}{d\tau} = 0 \quad \text{It is Geodesic equation} \quad \rightarrow \frac{De_0^\mu}{d\tau} = 0 \\
& \frac{De_1^\mu}{d\tau} = \frac{1}{u^0} \underbrace{\frac{Dl^\mu}{d\tau}}_{u^\alpha \nabla_\alpha l^\mu = 0} + \underbrace{\frac{Du_g^\mu}{d\tau}}_{=0} = 0 \\
& \frac{De_0^\mu}{d\tau} = \frac{De_1^\mu}{d\tau} = 0 \quad \text{they are parallel transported} \\
& \frac{De_2^\mu}{d\tau} \neq 0 \quad \frac{De_3^\mu}{d\tau} \neq 0
\end{aligned}$$

Clearly, tetrads 2 and 3 are not parallel transported. First, we construct new tetrads by rotating them by an angle ψ , and then we require that these tetrads be parallel transported along the geodesic.

$$\begin{aligned}
& \left\{ \begin{array}{l} e_2^\mu = \tilde{e}_2^\mu \cos \psi_{(U)} - \tilde{e}_3^\mu \sin \psi_{(U)} \\ e_3^\mu = \tilde{e}_2^\mu \sin \psi_{(U)} + \tilde{e}_3^\mu \cos \psi_{(U)} \end{array} \right\} \quad u_g^\nu \nabla_\nu e_2^\mu = u_g^\nu \nabla_\nu e_3^\mu = 0 \\
& \left\{ \begin{array}{l} u^\mu \left[\nabla_\nu \tilde{e}_2^\mu \cos \psi - \nabla_\nu \tilde{e}_3^\mu \sin \psi - \tilde{e}_2^\mu \sin \psi \frac{d\psi}{dU} \delta_{U,\nu} - \tilde{e}_3^\mu \cos \psi \frac{d\psi}{dU} \delta_{U,\nu} \right] = 0 \\ u^\mu \left[\nabla_\nu \tilde{e}_2^\mu \sin \psi + \nabla_\nu \tilde{e}_3^\mu \cos \psi + \tilde{e}_2^\mu \cos \psi \frac{d\psi}{dU} \delta_{U,\nu} - \tilde{e}_3^\mu \sin \psi \frac{d\psi}{dU} \delta_{U,\nu} \right] = 0 \end{array} \right\} \rightarrow u^\nu \nabla_\nu \tilde{e}_2^\mu = \tilde{e}_3^\mu \frac{d\psi}{dU} \delta_{(U,\nu)} u^\nu \\
& \Rightarrow \frac{d\psi}{dU} = u^\nu \frac{\tilde{e}_\mu^3 \nabla_\nu \tilde{e}_2^\mu}{u^0}
\end{aligned}$$

Now, we compute the following quantities

$$\begin{aligned}
u^\nu \nabla_\nu (\partial_y)^\mu &= u^\nu \Gamma_{\nu\alpha}^\mu (\partial_y)^\alpha = u^\nu \Gamma_{\nu 2}^\mu = \frac{u^\nu}{2} g^{\mu\alpha} (g_{\alpha\nu,2} + g_{2\alpha,\nu} - g_{\nu 2,\alpha}) \\
u^\nu \nabla_\nu (\partial_y)^\mu &= \frac{1}{2} (u^0 g^{\mu 2} g_{22,0} + u^0 g^{\mu 3} g_{23,0} - u^2 g^{\mu 0} g_{22,0} - u^3 g^{\mu 0} g_{32,0}) \\
u^\nu \nabla_\nu (\partial_y)^\mu &= \frac{1}{2} (-u^0 g^{\mu 2} \dot{h}_+ - u^0 g^{\mu 3} \dot{h}_\times + u^2 g^{\mu 0} \dot{h}_+ + u^3 g^{\mu 0} \dot{h}_\times)
\end{aligned}$$

Based on the above relations and Equation (9), we have

$$\begin{aligned}
\tilde{e}_\mu^3 u^\nu \nabla_\nu (\partial_y)^\mu &= \frac{1}{2} \sqrt{\frac{1-h_+}{-g}} \left[\frac{h_\times (\partial_y)_\mu}{1-h_+} + (\partial_z)_\mu + \frac{gu^3}{u^0(1-h_+)} (\partial_V)_\mu \right] \left[-u^0 g^{\mu 2} \dot{h}_+ - u^0 g^{\mu 3} \dot{h}_\times + u^2 g^{\mu 0} \dot{h}_+ + u^3 g^{\mu 0} \dot{h}_\times \right] \\
\tilde{e}_\mu^3 u^\nu \nabla_\nu (\partial_y)^\mu &= \frac{1}{2} \sqrt{\frac{1-h_+}{-g}} \left[\frac{-h_\times u^0 \dot{h}_+}{1-h_+} - u^0 \dot{h}_\times \right] = -\frac{u^0}{2} \sqrt{\frac{1-h_+}{-g}} \left[\frac{h_\times \dot{h}_+}{1-h_+} + \dot{h}_\times \right] \\
\frac{1}{u^0} \tilde{e}_\mu^3 u^\nu \nabla_\nu (\partial_y)^\mu &= -\frac{1}{2} \sqrt{\frac{1-h_+}{-g}} \left[\frac{h_\times \dot{h}_+}{1-h_+} + \dot{h}_\times \right]
\end{aligned}$$

Based on the Equation (8), we have

$$\stackrel{\nabla_\nu (\partial_V)^\mu = 0}{\rightarrow} \nabla_\nu \tilde{e}_2^\mu = \frac{\dot{h}_+ \delta_{U,\nu}}{2(1-h_+)^{\frac{3}{2}}} (\partial_y)^\mu + F_{(U)} (\partial_V)^\mu + \frac{1}{\sqrt{1-h_+}} \nabla_\nu (\partial_y)^\mu$$

where $F_{(U)}$ is a function that depends only on U .

$$\begin{aligned}
\tilde{e}_\mu^3 \nabla_\nu \tilde{e}_2^\mu &= \sqrt{\frac{1-h_+}{-g}} \left[\frac{\dot{h}_+ h_\times (1-h_+)}{2(1-h_+)^{\frac{3}{2}}} - \frac{\dot{h}_+ h_\times}{2(1-h_+)^{\frac{3}{2}}} \right] + \frac{1}{\sqrt{1-h_+}} \tilde{e}_\mu^3 \nabla_\nu (\partial_y)^\mu = \frac{1}{\sqrt{1-h_+}} \tilde{e}_\mu^3 \nabla_\nu (\partial_y)^\mu \\
\rightarrow \frac{u^\nu \tilde{e}_\mu^3 \nabla_\nu \tilde{e}_2^\mu}{u^0} &= \frac{u^\nu}{u^0 \sqrt{1-h_+}} \tilde{e}_\mu^3 \nabla_\nu (\partial_y)^\mu = \frac{-1}{2\sqrt{-g}} \left[\frac{h_\times \dot{h}_+}{1-h_+} + \dot{h}_\times \right]
\end{aligned}$$

$$\boxed{\frac{d\psi}{dU} = \frac{-1}{2\sqrt{-g}} \left[\frac{h_\times \dot{h}_+}{1-h_+} + \dot{h}_\times \right]}$$

10 Appendix D

$$C_1 := \frac{\dot{h}_+ h_\times + \dot{h}_\times (1 - h_+)}{2\sqrt{-g}} \quad C_2 := \frac{\dot{h}_+(1 + h_+) + \dot{h}_\times h_\times}{2\sqrt{-g}} \quad C_3 := \frac{\dot{h}_\times (1 + h_+) - \dot{h}_+ h_\times}{2\sqrt{-g}}$$

$$C_4 := \frac{\dot{h}_+(1 - h_+) - \dot{h}_\times h_\times}{2\sqrt{-g}}$$

Based on equations (2) and (3), we have

$$\begin{cases} J_I = (1 - h_+)P^y - h_\times P^z + C_1(P^y s^U - \varepsilon s^y) + C_2(\varepsilon s^z - P^z s^U) \\ J_{II} = (1 + h_+)P^z - h_\times P^y + C_3(-P^z s^U + \varepsilon s^z) + C_4(\varepsilon s^y - P^y s^U) \\ J_I = (1 - h_+ + C_1 s^U)P^y - (C_2 s^U + h_\times)P^z + \varepsilon(C_2 s^z - C_1 s^y) \\ J_{II} = -(C_4 s^U + h_\times)P^y + (1 + h_+ - C_3 s^U)P^z + \varepsilon(C_3 s^z + C_4 s^y) \end{cases} \quad (D0)$$

Now we define the following quantities

$$Q^y := \frac{J_I(1 + h_+) + J_{II}(h_\times)}{-g}$$

$$Q^z := \frac{h_\times J_I + (1 - h_+)J_{II}}{-g}$$

$$\left\{ \begin{array}{l} P^y = \frac{J_I(1+h_+)+J_{II}(h_\times)}{-g} + F_U^{(y)} s^U + F_y^{(y)} s^y + F_z^{(y)} s^z \\ P^z = \frac{h_\times J_I + (1-h_+)J_{II}}{-g} + F_U^{(z)} s^U + F_y^{(z)} s^y + F_z^{(z)} s^z \end{array} \right\}$$

Now we substitute the momenta from the two equations above into the set of Equations (D0)

$$\left\{ \begin{array}{l} J_I = (1 - h_+)Q^y + (1 - h_+) \left(F_U^{(y)} s^U + F_y^{(y)} s^y + F_z^{(y)} s^z \right) + C_1 Q^y s^U - h_\times Q^z \\ \quad - h_\times \left(F_U^{(z)} s^U + F_y^{(z)} s^y + F_z^{(z)} s^z \right) - C_2 s^U Q^z + \varepsilon(C_2 s^z - C_1 s^y) \\ J_{II} = -h_\times Q^y - h_\times \left(F_U^{(y)} s^U + F_y^{(y)} s^y + F_z^{(y)} s^z \right) - C_4 s^U Q^y + (1 + h_+)Q^z \\ \quad + (1 + h_+) \left(F_U^{(z)} s^U + F_y^{(z)} s^y + F_z^{(z)} s^z \right) - C_3 s^U Q^z + \varepsilon(C_3 s^z + C_4 s^y) \end{array} \right.$$

We calculate the following expressions

$$J_I - (1 - h_+)Q^y + h_\times Q^z = J_I - \frac{(1 - h_+^2)J_I + (1 - h_+)h_\times J_{II}}{-g} + \frac{h_\times^2 J_I + h_\times(1 - h_+)J_{II}}{-g}$$

$$= J_I + \frac{gJ_I + J_{II}(h_\times(1 - h_+) - h_\times(1 - h_+))}{-g} = J_I - J_I = 0 \quad \rightarrow$$

$$J_I - (1 - h_+)Q^y + h_\times Q^z = 0$$

$$J_{II} + h_\times Q^z - (1 + h_+)Q^y = J_{II} + \frac{h_\times(1 - h_+)J_I + h_\times^2 J_{II} - h_\times(1 + h_+)J_I - (1 - h_+^2)J_{II}}{-g} = J_{II} + \frac{gJ_{II}}{-g} = 0 \quad \rightarrow$$

$$J_{II} + h_\times Q^z - (1 + h_+)Q^y = 0$$

Now, by equating the coefficients of the independent components of the spin vector, we have

$$(1 - h_+)F_U^{(y)} + C_1Q^y - h_\times F_U^{(z)} - C_2Q^z = 0 \quad (D1)$$

$$(1 - h_+)F_y^{(y)} - h_\times F_y^{(z)} - \varepsilon C_1 = 0 \quad (D2)$$

$$(1 - h_+)F_z^{(y)} - h_\times F_z^{(z)} + \varepsilon C_2 = 0 \quad (D3)$$

$$-h_\times F_U^{(y)} - C_4Q^y + (1 + h_+)F_U^{(z)} = 0 \quad (D4)$$

$$-h_\times F_y^{(y)} + (1 + h_+)F_y^{(z)} + \varepsilon C_4 = 0 \quad (D5)$$

$$-h_\times F_z^{(y)} + (1 - h_+)F_z^{(z)} + \varepsilon C_3 = 0 \quad (D6)$$

$$\xrightarrow{(D1),(D4)} h_\times C_1Q^y - h_\times^2 F_U^{(z)} - C_2Q^z h_\times - C_4Q^y(1 - h_+) + (1 - h_+^2)F_U^{(z)} = 0$$

$$Q^y [h_\times C_1 - (1 - h_+)C_4] - C_2Q^z h_\times = F_U^{(z)} \underbrace{(h_\times^2 + h_+^2 - 1)}_g$$

$$\boxed{F_U^{(z)} = \frac{C_2Q^z h_\times + [-h_\times C_1 + (1 - h_+)C_4]Q^y}{-g}}$$

$$F_U^{(y)} = \frac{h_\times}{1 - h_+} F_U^{(z)} + \frac{-g(C_2Q^z - C_1Q^y)}{-g(1 - h_+)}$$

$$F_U^{(y)} = \frac{C_2Q^z h_\times^2 + h_\times((1 - h_+)C_4 - h_\times C_1)Q^y + C_1gQ^y - C_2gQ^z}{-g(1 - h_+)}$$

$$\boxed{F_U^{(y)} = \frac{C_2(h_\times^2 - g)Q^z + (h_\times(1 - h_+)C_4 - C_1(h_\times^2 - g))Q^y}{-g(1 - h_+)}}$$

$$\xrightarrow{(D2),(D5)} -h_\times^2 F_y^{(z)} - \varepsilon C_1 h_\times + (1 - h_+^2)F_y^{(z)} + \varepsilon C_4(1 - h_+) = 0$$

$$-gF_y^{(z)} + \varepsilon(C_4(1 - h_+) - C_1 h_\times) = 0$$

$$\boxed{F_y^{(z)} = \frac{\varepsilon(C_1 h_\times - C_4(1 - h_+))}{-g}}$$

$$F_y^{(y)} = \frac{\varepsilon C_1 + h_\times F_y^{(z)}}{1 - h_+} = \frac{\varepsilon C_1(-g)}{-g(1 - h_+)} + \frac{h_\times}{1 - h_+} \frac{\varepsilon(C_1 h_\times - C_4(1 - h_+))}{-g}$$

$$\boxed{F_y^{(y)} = \frac{\varepsilon(C_1(h_\times^2 - g) - C_4 h_\times(1 - h_+))}{-g(1 - h_+)}}$$

$$\xrightarrow{(D3),(D6)} -h_\times^2 F_z^{(z)} + \varepsilon C_2 h_\times + (1 - h_+^2)F_z^{(z)} + \varepsilon C_3(1 - h_+) = 0$$

$$-gF_z^{(z)} + \varepsilon C_2 h_\times + \varepsilon C_3(1 - h_+) = 0 \rightarrow$$

$$\boxed{F_z^{(z)} = \frac{-\varepsilon(C_2 h_\times + C_3(1 - h_+))}{-g}}$$

$$F_z^{(y)} = \frac{1 + h_+}{h_\times} F_z^{(z)} + \frac{\varepsilon C_3}{h_\times} = \frac{-\varepsilon(1 + h_+)(C_2 h_\times + C_3(1 - h_+)) - \varepsilon C_3 g}{-gh_\times}$$

$$F_z^{(y)} = \frac{\varepsilon}{gh_\times} [(1 + h_+)(C_2 h_\times + C_3(1 - h_+)) + C_3 g] = \frac{\varepsilon}{gh_\times} [(1 + h_+)h_\times C_2 + C_3 h_\times^2]$$

$$\boxed{F_z^{(y)} = \frac{-\varepsilon[(1 + h_+)C_2 + C_3 h_\times]}{-g}}$$

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